

A NOTE ON STRONGLY REAL BEAUVILLE p -GROUPS

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ABSTRACT. We give an infinite family of non-abelian strongly real Beauville p -groups for any odd prime p by considering the lower central quotients of the free product of two cyclic groups of order p . This is the first known infinite family of non-abelian strongly real Beauville p -groups.

1. INTRODUCTION

A *Beauville surface* of unmixed type is a compact complex surface isomorphic to $(C_1 \times C_2)/G$, where C_1 and C_2 are algebraic curves of genus at least 2 and G is a finite group acting freely on $C_1 \times C_2$ and faithfully on the factors C_i such that $C_i/G \cong \mathbb{P}_1(\mathbb{C})$ and the covering map $C_i \rightarrow C_i/G$ is ramified over three points for $i = 1, 2$. Then the group G is said to be a *Beauville group*.

The condition for a finite group G to be a Beauville group can be formulated in purely group-theoretical terms.

Definition 1.1. For a couple of elements $x, y \in G$, we define

$$\Sigma(x, y) = \bigcup_{g \in G} (\langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g),$$

that is, the union of all subgroups of G which are conjugate to $\langle x \rangle$, to $\langle y \rangle$ or to $\langle xy \rangle$. Then G is a Beauville group if and only if the following conditions hold:

- (i) G is a 2-generator group.
- (ii) There exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of G such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$.

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are said to form a *Beauville structure* for G .

Definition 1.2. Let G be a Beauville group. We say that G is *strongly real* if there exists a Beauville structure $\{\{x_1, y_1\}, \{x_2, y_2\}\}$ such that there exist an automorphism $\theta \in \text{Aut}(G)$ and elements $g_i \in G$ for $i = 1, 2$ such that

$$g_i \theta(x_i) g_i^{-1} = x_i^{-1} \quad \text{and} \quad g_i \theta(y_i) g_i^{-1} = y_i^{-1}$$

for $i = 1, 2$. Then the Beauville structure is called *strongly real Beauville structure*.

In practice, it is convenient to take $g_1 = g_2 = 1$.

In 2000, Catanese [3] proved that a finite abelian group is a Beauville group if and only if it is isomorphic to $C_n \times C_n$, where $n > 1$ and $\gcd(n, 6) = 1$.

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1. Since for any abelian group the function $x \mapsto -x$ is an automorphism, the following result is immediate.

Lemma 1.3. *Every abelian Beauville group is a strongly real Beauville group.*

Thus, there are infinitely many abelian strongly real Beauville p -groups for $p \geq 5$.

Recall that the only known infinite family of Beauville 2-groups was constructed in [1]. However, one of the main results in [1] shows that these Beauville 2-groups are not strongly real. On the other hand, in [5], Fairbairn has recently given the following examples of strongly real Beauville 2-groups. The groups

$$G = \langle x, y \mid x^8 = y^8 = [x^2, y^2] = (x^i y^j)^4 = 1 \text{ for } i, j = 1, 2, 3 \rangle,$$

and

$$G = \langle x, y \mid (x^i y^j)^4 = 1 \text{ for } i, j = 0, 1, 2, 3 \rangle$$

are strongly real Beauville groups of order 2^{13} and 2^{14} , respectively.

If $p \geq 3$ there is no known example of a non-abelian strongly real Beauville p -group. Thus, up to now the only examples of strongly real Beauville p -groups are the abelian ones and the two groups given above.

In this paper, we give infinitely many non-abelian strongly real Beauville p -groups for any odd prime p . To this purpose, we work with the lower central quotients of the free product of two cyclic groups of order p . The main result of this paper is as follows.

Theorem A. *Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order p for an odd prime p , and let $i = k(p-1) + 1$ for $k \geq 1$. Then the quotient $F/\gamma_{i+1}(F)$ is a strongly real Beauville group.*

Note that in [7], it was recently shown that all p -central quotients of the free product $F = \langle x, y \mid x^p, y^p \rangle$ are Beauville groups. Observe that since F/F' has exponent p , the lower central series and p -central series of F coincide.

2. PROOF OF THE MAIN THEOREM

In this section, we give the proof of Theorem A. Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order p . We begin by stating a lemma regarding the existence of an automorphism of F which sends the generators to their inverses. The proof is left to the reader.

Lemma 2.1. *Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups of order p . Then the map*

$$\begin{aligned} \theta: F &\longrightarrow F \\ x &\longmapsto x^{-1} \\ y &\longmapsto y^{-1}, \end{aligned}$$

is an automorphism of F .

Before we proceed, we will introduce some results regarding the Nottingham group which will help us to determine some properties of F .

The *Nottingham group* \mathcal{N} over the field \mathbb{F}_p , for odd p , is the (topological) group of normalized automorphisms of the ring $\mathbb{F}_p[[t]]$ of formal power series. For any positive integer k , the automorphisms $f \in \mathcal{N}$ such that $f(t) = t + \sum_{i \geq k+1} a_i t^i$ form an open normal subgroup \mathcal{N}_k of \mathcal{N} of index p^{k-1} . Observe that $|\mathcal{N}_k : \mathcal{N}_{k+1}| = p$ for all $k \geq 1$. We have the commutator formula

$$(1) \quad [\mathcal{N}_k, \mathcal{N}_\ell] = \begin{cases} \mathcal{N}_{k+\ell}, & \text{if } k \not\equiv \ell \pmod{p}, \\ \mathcal{N}_{k+\ell+1}, & \text{if } k \equiv \ell \pmod{p} \end{cases}$$

(see [2], Theorem 2). Thus the lower central series of \mathcal{N} is given by

$$(2) \quad \gamma_i(\mathcal{N}) = \mathcal{N}_{r(i)}, \quad \text{where } r(i) = i + 1 + \left\lfloor \frac{i-2}{p-1} \right\rfloor.$$

As a consequence, $|\gamma_i(\mathcal{N}) : \gamma_{i+1}(\mathcal{N})| \leq p^2$, and we have ‘diamonds’ of order p^2 if and only if i is of the form $i = k(p-1) + 1$ for some $k \geq 0$. In other words, the diamonds in the lower central series of \mathcal{N} correspond to quotients $\mathcal{N}_{kp+1}/\mathcal{N}_{kp+3}$.

Recall that by Remark 3 in [2], \mathcal{N} is topologically generated by the elements $a \in \mathcal{N}_1 \setminus \mathcal{N}_2$ and $b \in \mathcal{N}_2 \setminus \mathcal{N}_3$ given by $a(t) = t(1-t)^{-1}$ and $b(t) = t(1-2t)^{-1/2}$, which are both of order p .

In the following lemma, we need a result of Klopsch [8, formula (3.4)] regarding the centralizers of elements of order p of \mathcal{N} in some quotients $\mathcal{N}/\mathcal{N}_k$. More specifically, if $f \in \mathcal{N}_k \setminus \mathcal{N}_{k+1}$ is of order p , then for every $\ell = k+1+pn$ with $n \in \mathbb{N}$, we have

$$(3) \quad C_{\mathcal{N}/\mathcal{N}_\ell}(f\mathcal{N}_\ell) = C_{\mathcal{N}}(f)\mathcal{N}_{\ell-k}/\mathcal{N}_\ell.$$

Lemma 2.2. *Put $G = \mathcal{N}/\mathcal{N}_{kp+3}$ and $N_i = \mathcal{N}_i/\mathcal{N}_{kp+3}$ for $1 \leq i \leq kp+3$. If α is the image of a in G , then the set $\{[\alpha, g] \mid g \in G\}$ does not cover N_{kp+1} .*

Proof. To prove the lemma, we will show that $\{[\alpha, g] \mid g \in G\} \cap N_{kp+2} = 1$. Assume that $[\alpha, g] \in N_{kp+2}$ for some $g \in G$. Since $a \in \mathcal{N}_1 \setminus \mathcal{N}_2$ is of order p , it follows from (3) that

$$C_{\mathcal{N}/\mathcal{N}_{kp+2}}(a\mathcal{N}_{kp+2}) = C_{\mathcal{N}}(a)\mathcal{N}_{kp+1}/\mathcal{N}_{kp+2}.$$

Thus we can write $g = ch$, with $[\alpha, c] = 1$ and $h \in N_{kp+1}$. Then $[\alpha, g] = [\alpha, h] \in [G, N_{kp+1}] = 1$, since N_{kp+1} is central in G . \square

Lemma 2.3. *Put $H = F/\gamma_{i+1}(F)$, where $i = k(p-1) + 1$ for $k \geq 1$ and $H_i = \gamma_i(F)/\gamma_{i+1}(F)$. If u and v are the images of x and y in H , respectively, then the sets $\{[u, h] \mid h \in H\}$ and $\{[v, h] \mid h \in H\}$ do not cover H_i .*

Proof. Let $G = \mathcal{N}/\mathcal{N}_{kp+3}$, and let us call α and β the images of a and b in G , respectively. Since α and β are of order p and $\gamma_{i+1}(G) = 1$, the map

$$\begin{aligned} \psi: H &\longrightarrow G \\ u &\longmapsto \alpha \\ v &\longmapsto \beta, \end{aligned}$$

is well-defined and an epimorphism.

By Lemma 2.2, the set of commutators of α does not cover the subgroup $\gamma_i(G) = N_{kp+1}$. It then follows that the set $\{[u, h] \mid h \in H\}$ does not cover H_i . Since the roles of u and v are symmetric, we also conclude that the set $\{[v, h] \mid h \in H\}$ does not cover H_i , as desired. \square

To prove the main result, we need the following three lemmas.

Lemma 2.4. *Let $G = \langle a, b \rangle$ be a 2-generator p -group and $o(a) = p$, for some prime p . Then*

$$\left(\bigcup_{g \in G} \langle a \rangle^g \right) \cap \left(\bigcup_{g \in G} \langle b \rangle^g \right) = 1.$$

Proof. We assume that $x = (a^i)^g = (b^j)^h$ for some $i, j \in \mathbb{Z}$ and $g, h \in G$, and prove that $x = 1$. In the quotient $\overline{G} = G/\Phi(G) = \langle \overline{a} \rangle \times \langle \overline{b} \rangle$, we have $\overline{x} \in \langle \overline{a} \rangle \cap \langle \overline{b} \rangle = \overline{1}$ implying that $x \in \Phi(G)$. On the other hand, $x \in \langle a^g \rangle$, where a^g is of order p and $a^g \notin \Phi(G)$. It then follows that $x = 1$. \square

Lemma 2.5. [6, Lemma 3.8] *Let G be a finite p -group and let $x \in G \setminus \Phi(G)$ be an element of order p . If $t \in \Phi(G) \setminus \{[x, g] \mid g \in G\}$ then*

$$\left(\bigcup_{g \in G} \langle x \rangle^g \right) \cap \left(\bigcup_{g \in G} \langle xt \rangle^g \right) = 1.$$

Lemma 2.6. [7, Lemma 3.1] *Let $\psi: G_1 \rightarrow G_2$ be a group homomorphism, let $x_1, y_1 \in G_1$ and $x_2 = \psi(x_1)$, $y_2 = \psi(y_1)$. If $o(x_1) = o(x_2)$ then the condition $\langle x_2^{\psi(g)} \rangle \cap \langle y_2^{\psi(h)} \rangle = 1$ implies that $\langle x_1^g \rangle \cap \langle y_1^h \rangle = 1$ for $g, h \in G_1$.*

Let $H = F/\gamma_{i+1}(F)$ and let u and v be the images of x and y in H , respectively. In order to prove the main theorem, we need to know the order of uv . We first recall a result of Easterfield [4] regarding the exponent of $\Omega_j(G)$. More precisely, if G is a p -group, then for every $j, k \geq 1$, the condition $\gamma_{k(p-1)+1}(G) = 1$ implies that

$$(4) \quad \exp \Omega_j(G) \leq p^{j+k-1}.$$

If we set $k = \left\lceil \frac{i}{p-1} \right\rceil$, we have $\gamma_{k(p-1)+1}(H) \leq \gamma_{i+1}(H) = 1$. Then by (4), we get $\exp H \leq p^k$, and hence $o(uv) \leq p^k$. Indeed, we will show that $o(uv) = p^k$. To this purpose, we also need to introduce a result regarding p -groups of maximal class with some specific properties.

Let $G = \langle s \rangle \rtimes A$ where s is of order p and $A \cong \mathbb{Z}_p^{p-1}$. The action of s on A is via θ , where θ is defined by the companion matrix of the p th cyclotomic polynomial $x^{p-1} + \cdots + x + 1$. Then G is the only infinite pro- p group of maximal class. Since $s^p = 1$ and $\theta^{p-1} + \cdots + \theta + 1$ annihilates A , this implies that for every $a \in A$,

$$(sa)^p = s^p a^{s^{p-1} + \cdots + s + 1} = 1.$$

Thus all elements in $G \setminus A$ are of order p .

Let P be a finite quotient of G of order p^{i+1} for $i \geq 2$. Let us call P_1 the abelian maximal subgroup of P and $P_j = [P_1, P, \overset{j}{\underbrace{\cdot, \cdot, \cdot}_{j-1}}, P] = \gamma_j(P)$ for

$j \geq 2$. Then one can easily check that $\exp P_j = p^{\lceil \frac{i+1-j}{p-1} \rceil}$ and every element in $P_j \setminus P_{j+1}$ is of order $p^{\lceil \frac{i+1-j}{p-1} \rceil}$.

Let $s \in P \setminus P_1$ and $s_1 \in P_1 \setminus P'$. Since all elements in $P \setminus P_1$ are of order p and $\gamma_{i+1}(P) = 1$, the map

$$\begin{aligned} \psi: H &\longrightarrow P \\ u &\longmapsto s^{-1} \\ v &\longmapsto ss_1, \end{aligned}$$

is well-defined and an epimorphism. Then we have $o(uv) \geq o(s_1) = p^k$, and this, together with $\exp H = p^k$, implies that $o(uv) = p^k$.

We are now ready to give the proof of main theorem.

Theorem 2.7. *Let $p \geq 3$, and let $i = k(p-1) + 1$ for $k \geq 1$. Then the quotient $F/\gamma_{i+1}(F)$ is a strongly real Beauville group.*

Proof. Let H and H_i be as defined in Lemma 2.3. Let u and v be the images of x and y in H , respectively. By Lemma 2.3, there exist $w, z \in H_i$ such that $w \notin \{[u, h] \mid h \in H\}$ and $z \notin \{[v, h] \mid h \in H\}$. Observe that w and z are central elements of order p in H . We claim that $\{u, v\}$ and $\{(uw)^{-1}, vz\}$ form a Beauville structure in H . Let $X = \{u, v, uv\}$ and $Y = \{(uw)^{-1}, vz, u^{-1}vw^{-1}z\}$.

Assume first that $x \in X$ is of order p , and let $y \in Y$. If $\langle x\Phi(H) \rangle \neq \langle y\Phi(H) \rangle$ in $H/\Phi(H)$, then by Lemma 2.4, $\langle x \rangle^g \cap \langle y \rangle^h = 1$ for every $g, h \in H$. Otherwise, we are in one of the following two cases: $x = u$ and $y = (uw)^{-1}$, or $x = v$ and $y = vz$. Then the condition $\langle x \rangle^g \cap \langle y \rangle^h = 1$ follows by Lemma 2.5.

We now assume that $x = uv$. Again applying Lemma 2.4, we get $\langle x \rangle^g \cap \langle y \rangle^h = 1$ where $y = (uw)^{-1}$ or $y = vz$, which is of order p . Thus we are only left with the case when $x = uv$ and $y = u^{-1}vw^{-1}z$. Recall that the map $\psi: H \longrightarrow P$ is an epimorphism such that $\psi(u) = s^{-1}$ and $\psi(v) = ss_1$. Then $\psi(u^{-1}vw^{-1}z)$ is an element outside P_1 , which is of order p . Thus $\langle \psi(u^{-1}vw^{-1}z)^{\psi(g)} \rangle \cap \langle s_1^{\psi(h)} \rangle = 1$ for all $g, h \in H$. Since $o(uv) = o(s_1)$, the condition $\langle x \rangle^g \cap \langle y \rangle^h = 1$ for all $g, h \in H$ follows by Lemma 2.6. This completes the proof that G is a Beauville group.

We next show that the Beauville structure $\{\{u, v\}, \{(uw)^{-1}, vz\}\}$ is strongly real. By Lemma 2.1, we know that the map θ is an automorphism of F . Since $\theta(\gamma_n(F)) = \gamma_n(\theta(F)) = \gamma_n(F)$ for all $n \geq 1$, the map θ induces an automorphism $\bar{\theta}: H \longrightarrow H$ such that $\bar{\theta}(u) = u^{-1}$ and $\bar{\theta}(v) = v^{-1}$. Now we only need to check if $\bar{\theta}((uw)^{-1}) = uw$ and $\bar{\theta}(vz) = (vz)^{-1}$. Note that

$$\bar{\theta}((uw)^{-1}) = \bar{\theta}(w^{-1})u = u\bar{\theta}(w^{-1}),$$

and

$$\bar{\theta}(vz) = v^{-1}\bar{\theta}(z) = \bar{\theta}(z)v^{-1}$$

where the last equalities follow from the fact that both w and z are central in H . Thus it suffices to see that $\bar{\theta}(w^{-1}) = w$ and $\bar{\theta}(z) = z^{-1}$.

Note that H_i is generated by the commutators of length i in u and v . Since i is odd and $H_i \leq Z(G)$, it follows that

$$\bar{\theta}([x_{j_1}, x_{j_2}, \dots, x_{j_i}]) = [x_{j_1}^{-1}, x_{j_2}^{-1}, \dots, x_{j_i}^{-1}] = [x_{j_1}, x_{j_2}, \dots, x_{j_i}]^{-1},$$

where each x_{j_k} is either u or v . Hence the automorphism $\bar{\theta}$ sends the generators of H_i to their inverses. Since H_i is abelian, this implies that for every $t \in H_i$ we have $\bar{\theta}(t) = t^{-1}$. \square

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